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What do CFTs tell us about anti-de Sitter spacetimes?

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Abstract: The AdS/CFT conjecture relates quantum gravity on anti-de Sitter (AdS) space to a conformal field theory (CFT) defined on the spacetime boundary. We interpret the CFT in terms of natural analogues of the bulk S-matrix. Our first approach finds the bulk S-matrix as a limit of scattering from an AdS bubble immersed in a space admitting asymptotic states. Next, we show how the periodicity of geodesics obstructs a standard LSZ prescription for scattering within global AdS. To avoid this subtlety we partition global AdS into patches within which CFT correlators reconstruct transition amplitudes of AdS states. Finally, we use the AdS/CFT duality to propose a large N collective field theory that describes local, perturbative supergravity. Failure of locality in quantum gravity should be related to the difference between the collective $1/N$ expansion and genuine finite N dynamics.

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Contents

1. Introduction

The AdS/CFT correspondence states that string theory in anti-de Sitter (AdS) spacetime is "holographically" dual to a conformal field theory (CFT) defined on the spacetime boundary [1, 2, 3]. The semiclassical limit of spacetime physics is related to the large N limit of the CFT. Semiclassical spacetime physics is generally expected to satisfy the familiar axioms of locality and causality. On the other hand, some sort of violation of locality is expected for a holographic quantum gravity that satisfies the Bekenstein bound [4]. For example, anti-de Sitter spacetime locality appears to translate into scale locality in the dual CFT, a property that is certainly absent for finite N [5, 6, 7]. Given holography, we are therefore faced, nearly inevitably, with violations of locality in semiclassical quantum gravity.

To investigate the failure of locality, we must understand what data about spacetime physics are accessible from CFT calculations. As summarized in sec. 2, the AdS/CFT correspondence equates the anti-de Sitter effective action, seen as a functional of boundary data, to the generating functional for correlators of a CFT defined on the spacetime boundary [2, 3]. All information available in the CFT is contained in these correlation functions, evaluated in all the possible states of the theory. We will show that these data reconstruct bulk transition amplitudes.

In AdS spacetimes of lorentzian signature, solutions to the free bulk wave equations can be classified into "normalizable" and "non-normalizable" modes encoding, respectively, the states of the theory and the boundary conditions for fields [8]. A conventional LSZ prescription for spacetime physics would describe transition amplitudes between asymptotic states, in terms of truncated Green functions integrated against normalizable modes. AdS lore states that asymptotic states cannot be defined, due to the timelike boundary and the periodicity of geodesics. However, in sec. 3 we will show that the vacuum correlation functions of the dual CFT can be given an S-matrix interpretation as a limit of scattering from an AdS "bubble" immersed in a space admitting asymptotic states. In Poincaré coordinates for AdS_5, this is simply the usual CFT interpretation of scattering from an asymptotically flat 3-brane [9, 10]. Global AdS does not arise this way as a limit of a conventional brane solution. Therefore, for interpretational purposes, we construct a global AdS bubble inside a simple asymptotic spacetime. A stable string solution of this kind is not known. Nevertheless, by studying scattering in this spacetime, we will argue that the CFT correlators recover universal features of an S-matrix for scattering in global AdS. Interestingly, our interpretation identifies certain singularities of the CFT correlators with resonant scattering from the anti-de Sitter region.

The results of sec. 3 arise because CFT correlators are expressed in the bulk as truncated Green functions convolved against non-normalizable modes. In sec. 4 we return to the issue of normalizable modes, or states localized *within* AdS space, and transitions between them. We start by showing explicitly that the periodicity of geodesics in AdS obstructs a conventional definition of asymptotic states. Thus the LSZ prescription is ill-defined. Instead, we partition global AdS into Poincaré patches within which geodesics do not reconverge. The AdS boundary is likewise partitioned, and physics within a given bulk patch is dual to a CFT defined on the corresponding boundary. The in and out states of the theory on the patch correspond to boundary conditions at early and late times. We show that transition amplitudes between these states are described by correlation functions of the dual CFT. In the large radius limit for AdS, this construction provides a holographic description of a flat space S-matrix. (While this paper was being written we received [11, 12] which contain a related derivation.) The intermediate steps of our discussion rely on the diagrammatic expansion of the spacetime physics, but we expect that the final results are defined in the full theory.

In sec. 5 we attempt to reconstruct local bulk operators from the CFT. After reviewing the $N \to \infty$ case where this reconstruction is manifest [6, 7], we discuss the problem at finite N. Using a Lehmann representation described by Düsedau and Freedman [13], we show that finite N bulk operators have a complicated "multi-particle" structure and that the CFT primaries only capture the "single-particle" piece. To reconstruct supergravity, we propose a "collective field theory" built from the spectral decomposition of large-N conformal primaries in the CFT. We are able to posit an effective action order by order in $1/N$ and the string coupling which reproduces local, perturbative spacetime physics after a simple Bessel transformation of collective field correlators. Presumably this collective field procedure reproduces the finite N CFT at best in a power series. We close with a discussion of the problems facing reconstruction of local supergravity from the exact finite N dynamics.

2. Review of the AdS/CFT correspondence

Euclidean AdS (EAdS) space is topologically a ball, with the metric:

$$ds^2 = \frac{R^2}{z^2}(dt^2 + d\vec{x}^2 + dz^2)\,. \tag{2.1}$$

Here $\mathbf{b} \equiv \{t, \vec{x}\}$ spans $\mathbb{R}^d$ and $\infty \leq z \leq 0$ with a boundary at $z = 0$. The EAdS/CFT correspondence states [2, 3]:

$$Z_{\text{bulk}}[\phi(\phi_0)] = \langle e^{-\int_\partial d^d\mathbf{b}\,\phi_0(\mathbf{b})\,\mathcal{O}(\mathbf{b})}\rangle\,. \tag{2.2}$$

Here $\ln Z_{\text{bulk}}$ is the effective action for string theory on AdS_{d+1} considered as a functional of the boundary data ϕ_0 for the fields ϕ. The right hand side is the generating functional of correlators of the operator $\mathcal{O}$ dual to ϕ. For illustration, we will always take ϕ to be a scalar field of mass m. Then, setting

$$2h_\pm = \frac{d}{2} \pm \nu\,,; \qquad \nu = \frac{1}{2}\sqrt{d^2 + 4m^2}\,, \tag{2.3}$$

regular classical solutions of the free wave equation for ϕ have the boundary behaviour

$$\lim_{z\to 0} \phi(\mathbf{b}, z) = z^{2h_-}\phi_0(\mathbf{b})\,, \tag{2.4}$$

and couple to CFT operators $\mathcal{O}$ of dimension $2h_+$. The free classical solution may be suggestively written in terms of the boundary value ϕ_0 and a bulk-boundary propagator $G^E_{B\partial}$ as [3]:

$$\phi(\mathbf{x}) = \int_\partial d\mathbf{b}\, G^E_{B\partial}(\mathbf{x}, \mathbf{b})\,\phi_0(\mathbf{b})\,, \tag{2.5}$$

with $\mathbf{x} \equiv \{z, t, \vec{x}\}$. Using this expression and (2.2), the CFT correlators are given in terms of truncated bulk Green functions as:

$$\langle \mathcal{O}(\mathbf{b}_1)\mathcal{O}(\mathbf{b}_2)\cdots\mathcal{O}(\mathbf{b}_n)\rangle = \int \prod_{i=1}^{n} \left[dx_i\, G^E_{B\partial}(\mathbf{b}_i, \mathbf{x}_i)\right] \langle \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle_T \,. \tag{2.6}$$

The EAdS diagrams introduced in [3] summarize this computation. In fig. 1 a diametric slice of EAdS is displayed, and the thick line is the boundary of the resulting disc. The CFT correlators are then obtained by replacing the legs of ordinary bulk Feynman diagrams by bulk-boundary propagators.[1]

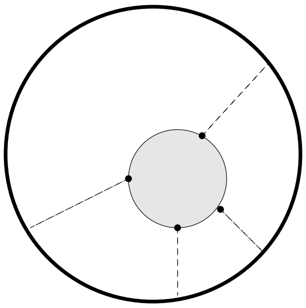

Figure 1: EAdS diagrams computing euclidean CFT correlators.

Lorentzian correspondence. The universal cover of lorentzian AdS (CAdS) is topologically a cylinder, with the metric:

$$ds^2 = R^2(-\sec^2\rho\, d\tau^2 + \sec^2\rho\, d\rho^2 + \tan^2\rho\, d\Omega^2_{d-1})\,. \tag{2.7}$$

The boundary of spacetime at $\rho = \pi/2$ has the topology $S^{d-1} \times R$. A free massive scalar field on this background will have regular solutions of the form:

$$\phi(x) = \int d\mathbf{b}\, G_{B\partial}(\mathbf{x}, \mathbf{b})\, \phi_0(\mathbf{b}) + \phi_n(\mathbf{x})\,, \tag{2.8}$$

where ϕ_n is normalizable in the Klein-Gordon norm and vanishes at the spacetime boundary. The bulk-boundary propagator $G_{B\partial}$ is a solution to the bulk wave equation which approaches a delta function on the AdS boundary. This propagator is ambiguous in lorentzian AdS since we may always add a normalizable solution ϕ_n to it without changing the boundary behaviour. We will pick the propagator arising via continuation from euclidean AdS. This corresponds to a certain choice of vacuum as discussed in sec. 3. Then (2.8) describes the corresponding general bulk solution that approaches ϕ_0 at the boundary.

It was argued in [8] that the normalizable, fluctuating solutions ϕ_n encode the states of the bulk subject to fixed boundary conditions specified by ϕ_0. The solutions ϕ_n provide a unitary representation of the conformal group that matches the states we expect the dual operator to create in the boundary CFT. In the free limit, *classical* bulk modes ϕ_n should be dual to "coherent" states in the boundary. In this limit the CAdS/CFT correspondence is written as [7]:

$$Z_{\rm cl}[\phi(\phi_0) + \phi_n] = \left\langle \phi_n | e^{i\int_\partial d\mathbf{b}\,\phi_0(\mathbf{b})\,\mathcal{O}(\mathbf{b})} | \phi_n \right\rangle, \tag{2.9}$$

where the bulk action is evaluated in the presence of the classical mode ϕ_n and the CFT is placed in corresponding "coherent" state $|\phi_n\rangle$. In this limit we do not

[1] Ref. [6] and section 2.1 show that they can also be obtained from the limiting behavior of bulk diagrams as the external points are taken to the boundary.

interpret the computation diagramatically and instead (semiclassically) evaluate the bulk action by using the equations of motion. It was shown in [7] that the classical "probe" ϕ_n induces expectation values for operators in the dual CFT.

Here we are interested in the interpretation of the normalizable modes as asymptotic states in transition amplitudes for the bulk theory. To have such an interpretation we must either be able to turn off bulk interactions at early and late times or separate the wavepackets by large distances. Since the effective bulk coupling constant is determined by the constant string coupling and AdS curvature, we are unable to change the asymptotic strength of the interaction. But in CAdS it is also not possible to separate wavepackets asymptotically because of the periodicity of geodesics in spacetime. Alternatively, the normalizable mode solutions of CAdS have fixed temporal periodicity [14, 15, 16, 7]. So, if two wavepackets interact in a near collision and and then appear to separate, they eventually bounce off the AdS effective potential near the boundary and almost collide again. On the other hand, there should be states in the bulk theory that are dual to the CFT states that arise from operators acting on the vacuum at early and late times. We will discuss these issues in detail in sec. 4.

For the moment, we will work in the CAdS vacuum by setting all the modes ϕ_n to zero. Then the CAdS/CFT correspondence is given by:

$$Z[\phi(\phi_0)] = \left\langle T\, e^{i\int_\partial d\mathbf{b}\, \phi_0(\mathbf{b})\, \mathcal{O}(\mathbf{b})} \right\rangle , \tag{2.10}$$

and the CFT vacuum correlators are expressed in terms of truncated bulk Green functions as:

$$\langle T\, \mathcal{O}(\mathbf{b}_1)\mathcal{O}(\mathbf{b}_2)\cdots\mathcal{O}(\mathbf{b}_n)\rangle = \int \prod_{i=1}^{n} \left[d\mathbf{x}_i\, G_{B\partial}(\mathbf{b}_i, \mathbf{x}_i)\right] \langle T\, \phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\rangle_T\, . \tag{2.11}$$

The corresponding CAdS diagram appears in fig. 2a. Once again, a diametric slice of CAdS is presented and the thick lines represent the cylindrical AdS boundary. In sec. 4 we will discuss how (2.10) is modified to account for the states of AdS and transitions between them. Essentially, this will result in CAdS diagrams with extra legs (fig. 2b, for example) representing the influence of in and out states on CFT correlators.

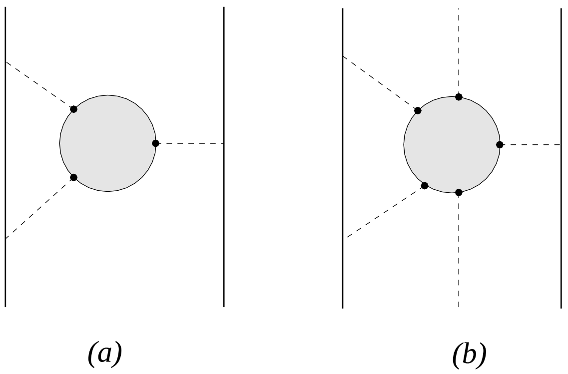

Figure 2: CAdS diagrams computing lorentzian correlators.

2.1 The bulk-boundary propagator

We are interested in determining what data about the bulk spacetime are contained in the CFT correlators in (2.11). The first step is to determine the bulk-boundary propagator in lorentzian spacetimes.

Euclidean propagator. The euclidean bulk-boundary propagator was defined in [3] to be a solution of the bulk wave equation that approaches a delta function at the boundary.[2] It is more convenient for us to Fourier transform with respect to the boundary coordinates $\mathbf{b}$. Then, $G_{B\partial}(\mathbf{k},\mathbf{x})$ must be a solution to the bulk wave equation that is non-singular in the bulk and is asymptotic to a plane wave at the $z=0$ boundary. This gives [17], [7]:

$$G_{B\partial}(\mathbf{k},\mathbf{x}) = A\, z^{d/2}\, K_\nu(qz)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\,. \tag{2.12}$$

Here A is a normalization factor, $\mathbf{k} \equiv \{\omega, \vec{k}\}$ and $q^2 = \omega^2 + \vec{k}^2$. The Bessel function K_ν vanishes exponentially as $z \to \infty$ and scales as $z^{-\nu}$ at the boundary as $z \to 0$.

Poincaré propagator. We arrive at the Poincaré patch (PAdS) of CAdS by performing the Wick rotation $t \to it$ in (2.1):

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2)\,. \tag{2.13}$$

Fig. 3a displays a diametric slice of CAdS as an infinite tower of PAdS patches.[3] Poincaré observers in each patch see past and future horizons (H^-, H^+) at $z=\infty$, where the patches meet. The boundary B at $z=0$ of a given patch is conformal to the Minkowski plane. The PAdS/CFT duality relates physics within the Poincaré patch to a CFT defined on the planar boundary.[4] The PAdS propagator appearing in (2.11) is then obtained by continuation from (2.12). Setting $q^2 = \omega^2 - \vec{k}^2$ we find:

$$G_{B\partial}(\mathbf{k},\mathbf{x}) = \tilde{A}\, z^{d/2}\, H^{(1)}_\nu(qz)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}\,. \tag{2.14}$$

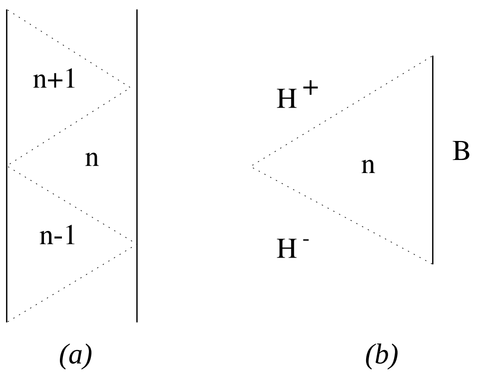

Figure 3: Constructing CAdS from PAds. (a) CAdS as a PAdS tower. (b) Poincaré AdS.

[2] In practice, calculations are carried out by cutting off EAdS at $z=\epsilon$ and removing this regulator at the end. In order to obey the CFT Ward identities we should actually require $G_{B\partial}$ to approach a delta function at $z=\epsilon$ [17] .

[3] See the appendix of [8] for more details.

[4] A CFT strictly speaking cannot be defined on the Minkowski plane. An infinite stack of planes is needed to realize global conformal transformations [18] . This is dual in spacetime to the realization of the global isometries of CAdS on an infinite tower of Poincaré patches.

Here H_ν is a Hankel function, $q = \sqrt{q^2}$ and for spacelike momenta ($q^2 < 0$) we choose the branch $q = i\sqrt{|q^2|}$. Then, for timelike momenta and positive ω, $G_{B\partial}$ is purely ingoing at the horizon while for $q^2 < 0$, $G_{B\partial}$ vanishes exponentially at the horizon. For timelike momenta there is also a spectrum of normalizable mode solutions proportional to $J_\nu(qz)$ [8] . Such modes may be added to the bulk-boundary propagator, inducing an outgoing component at the past horizon. The propagator (2.14) corresponds to a choice of Hartle-Hawking vacuum as usual (see e.g. [19]) upon taking a euclidean continuation.

The PAdS bulk-boundary propagator may also be derived by taking a scaling limit of the bulk Feynman propagator [6] :

$$G_{B\partial}(\mathbf{b}; \mathbf{b}', z') = \lim_{z\to 0} z^{-2h_+} G_F(\mathbf{b}, z; \mathbf{b}', z') \,. \tag{2.15}$$

To show this we explicitly construct the bulk Feynman propagator from the normalizable modes given in [8] and find:[5]

$$\tilde{G}_{B\partial}(\mathbf{b}_1; \mathbf{b}_2, z_2) = \int_{q\geq 0} dq \left(\frac{q}{2}\right)^{1+\nu} \frac{1}{\Gamma(1+\nu)} z_2^{d/2} J_\nu(qz_2) \int \frac{d\vec{k}d\omega}{(2\pi)^d} \frac{e^{-i\omega(t_1-t_2)+i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{-\omega^2+\vec{k}^2+q^2-i\epsilon} \,. \tag{2.16}$$

Fourier transforming and using Bessel function identities to convert the integral over $q \geq 0$ into an integral over the real line gives:

$$\tilde{G}_{B\partial}(\mathbf{k}; z, \mathbf{b}) = \frac{q^\nu \, z^{d/2}}{2^{2+\nu}\,\Gamma(1+\nu)} \, H_\nu^{(1)}(qz) \, e^{i\mathbf{k}\cdot\mathbf{b}} \,, \tag{2.17}$$

in agreement with (2.14). This relation between the bulk Feynman propagator and the bulk-boundary propagator will be useful to us in developing an S-matrix interpretation of CFT correlators.

CAdS propagator. Once again, in global AdS we seek a solution to the wave equation approaching the delta function at the boundary [3, 17]. After expanding in spherical harmonics on the boundary sphere we require a smooth solution with the boundary behaviour

$$G_{B\partial} \to e^{-i\omega\tau} Y_{l\vec{m}}(\Omega)(\cos\rho)^{2h_-} \tag{2.18}$$

as $\rho \to \pi/2$, where $Y_{l\vec{m}}$ are the spherical harmonics (cf. [20, 21]). The unique solution for fixed $\{\omega, l, \vec{m}\}$ is (see e.g. [8]):

$$\begin{aligned} G_{B\partial}(\omega, l, \vec{m}; \tau, \Omega, \rho) = {} & B\, e^{-i\omega\tau}\, Y_{l,\vec{m}}(\Omega)(\cos\rho)^{2h_+}(\sin\rho)^l \times \\ & \times {}_2F_1\left(\frac{1}{2}(2h_+ + l + \omega), \frac{1}{2}(2h_+ + l - \omega), l + \frac{d}{2}, \sin^2\rho\right), \end{aligned} \tag{2.19}$$

where B is a normalization constant. For any given l, at the frequencies $\omega = 2h_+ + l + 2n$ with $n = \{0, 1, \ldots\}$, there is a spectrum of normalizable modes that vanish

[5]Here it is important to correctly normalize the bulk modes to 1 in the Klein-Gordon norm.

at the boundary. These may be added to $G_{B\partial}$ without changing the boundary behaviour. To eliminate the ambiguity, consider the euclidean continuation $\tau \to i\tau$ of the CAdS metric (2.7). The bulk-boundary propagator in that metric is a smooth solution to an elliptic differential equation and is unique given a fixed boundary behaviour (2.18). Continuing back to lorentzian signature does not mix spherical harmonics or frequencies, giving (2.19) as the propagator.

The CAdS propagator (2.19) has singularities at the frequencies $\omega = 2h_+ + l + 2n$ corresponding to normalizable spacetime states. This is a result of the normalization condition (2.18). To see this, rewrite the hypergeometric function ${}_2F_1$ as a function of $\cos^2\rho$ following [8]:

$$(\cos\rho)^{2h_+}\,(\sin\rho)^l\,F_1(\sin^2\rho) = C^+\,\Phi^{(+)} + C^-\,\Phi^{(-)}\,, \tag{2.20}$$

where $\Phi^{(+)}$ vanishes at the boundary and $\Phi^{(-)}$ achieves the asymptotics (2.18). The coefficients $C^\pm$ are given by:[6]

$$C^\pm = \frac{\Gamma(l+\frac{d}{2})\Gamma(\mp\nu)}{\Gamma\left(\frac{1}{2}(2h_\mp + l + \omega)\right)\Gamma\left(\frac{1}{2}(2h_\mp + l - \omega)\right)}\,. \tag{2.21}$$

We maintain the asymptotics (2.18) by picking $B = 1/C_-$. But C_- vanishes when

$$\omega = \omega_{nl} = 2h_+ + l + 2n\,, \qquad n = 0, 1, 2, \ldots \tag{2.22}$$

giving a divergence in $G_{B\partial}$. The divergence is proportional to $\Phi^{(+)}$ which, at the magic frequencies (2.22), are precisely the normalizable states of the bulk theory. In later sections, we will interpret these singularities in terms of resonant scattering behaviour.

As in Poincaré coordinates, $G_{B\partial}$ can be derived as a limit of the bulk Feynman propagator in terms of the complete set of modes derived in e.g. [8]:

$$\Phi^+_{nl\vec{m}}(\rho,\Omega) = N_{nl}\,(\cos\rho)^{2h_+}\,(\sin\rho)^l\;P_n^{(l-1+d/2,2h_+-d/2)}(\cos 2\rho)\;Y_{l\vec{m}}(\Omega)\,, \tag{2.23}$$

where N_{nl} is a normalization factor and $P_n^{k,m}$ are Jacobi polynomials. These modes have quantized frequencies ω_{nl} given in (2.22). The Feynman propagator is then:

$$G_B(\mathbf{x}_1,\mathbf{x}_2) = \langle T\,\Phi(\mathbf{x}_1)\Phi(\mathbf{x}_2)\rangle = \int_{-\infty}^{\infty} d\omega \sum_{n,l,\vec{m}} \frac{\Phi^{+*}_{nl\vec{m}}(\rho_1,\Omega_1)\Phi^{+}_{nl\vec{m}}(\rho_2,\Omega_2)e^{i\omega(\tau_1-\tau_2)}}{\omega^2_{n,l} - \omega^2 - i\epsilon}\,. \tag{2.24}$$

We find the bulk-boundary propagator by taking one argument to the boundary and rescaling:

$$G_{B\partial}(t_1,\Omega_1;\mathbf{x}_2) = \lim_{\rho_1\to\pi/2}(\cos\rho_1)^{-2h_+}G_B(\mathbf{x}_1,\mathbf{x}_2)\,. \tag{2.25}$$

[6] For $\nu \in \mathbb{Z}$ the divergent Gamma function in the numerator of C^+ will be cancelled by a pole in the denominator of $\Phi^{(+)}$.

Projecting the resulting propagator onto definite frequencies and angular momenta gives:

$$G_{B\partial}(\omega, l, \vec{m}; \mathbf{x}) = \sum_n N'_{n,l} e^{-i\omega\tau} \frac{\Phi^+_{nl\vec{m}}(\rho, \Omega)}{\omega_{nl}^2 - \omega^2 - i\epsilon}\,, \tag{2.26}$$

where N' is a normalization factor. Note that when $\omega \to \omega_{nl}$, the n^{th} term in the sum dominates with an infinite coefficient. We then get a pole factor times $\Phi^+_{nl\vec{m}}$, replicating the resonant behaviour described above.

3. Scattering from an AdS bubble

We are now prepared to interpret the CFT correlators from the spacetime perspective. Begin by Fourier transforming the vacuum correlators of the CFT in (2.11). Let $\psi_\alpha(\mathbf{x})$ be the transformed bulk-boundary propagator with $\alpha \equiv \mathbf{k} = \{\omega, \vec{k}\}$ in Poincaré coordinates, and $\alpha \equiv \{\omega, l, \vec{m}\}$ in CAdS. Then (2.11) becomes:

$$\langle T\, \mathcal{O}_1(\alpha_1) \cdots \mathcal{O}_n(\alpha_n)\rangle = \int \prod_{k=1}^{n} d\mathbf{x}_k\, \psi_{\alpha_1}(\mathbf{x}_1) \cdots \psi_{\alpha_n}(\mathbf{x}_n)\, \langle T\, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n)\rangle_{\mathrm{T}}\,. \tag{3.1}$$

The left hand side of this expression is simply the Fourier transform of a CFT correlator. Here ψ_α is given by the mode solutions (2.14) and (2.19) in PAdS and CAdS respectively. So the right side of (3.1) looks just like an LSZ formula for an S-matrix element: it is a truncated bulk Green function, with its legs projected onto on-shell wavefunctions.

Nevertheless (3.1) is not giving us a conventional LSZ prescription for AdS states. As observed in sec. 2, the notion of asymptotic states in CAdS in problematic because of the reconvergence of geodesics. We will make this observation precise in sec. 4 and show that conventional transition amplitudes may only be defined for suitable trucations of CAdS. Regardless, the modes ψ_α appearing in (3.1) are simply not the states of the theory. For general ω these modes have infinite action and do not fluctuate, a property related to their divergence near the infinite volume AdS boundary [8].[7]

The expansion of the bulk-boundary propagator in terms of normalizable modes provides an important clue to the spacetime interpretation of (3.1). We will argue in sec. 4 that a disturbance created at the AdS boundary at a given time can be resolved into a sum of normalizable modes at a later time. This is dual to a statement that propagating states are produced in the CFT by operators acting at early times. In sec. 4 we will use this fact to interpret (3.1) in terms of a transition amplitude for states in a temporal AdS "box" (a temporally truncated anti-de Sitter patch).

[7]Actually, in CAdS ψ_α is normalizable at the magic frequencies (2.22), but, as discussed in sec. 2.1, it has a divergent coefficient and cannot be thought of as an AdS state.

Here we pursue a complementary interpretation in terms of a spatial AdS bubble. The bulk-boundary propagator is explicitly constructed to transport the influence of disturbances of the AdS boundary into the interior of the bubble. So (3.1) summarizes the response of the AdS bubble to measurements by an observer external to the bubble. More generally, imagine a large bubble of AdS space inside an asymptotically flat spacetime. (The bubble might be metastable and could eventually dissipate.) A strong form of the holographic proposal says that quantum gravity inside any volume should be described by a theory living on the boundary of the volume [4]. We then expect the large AdS bubble to be described by a CFT living on its boundary. An experimentalist probing the bubble can make widely separated wavepackets and focus them to enter the bubble, where they interact. The S-matrix for this scattering process should be described by correlation functions of the CFT. Of course, the fields would have to be suitably normalized to reflect the probability that the wavepackets penetrate the AdS region. But the non-trivial part of the scattering process is precisely summarized by (3.1). We will develop examples of this "bubble" interpretation, displayed in fig. 4, in the remainder of this section.

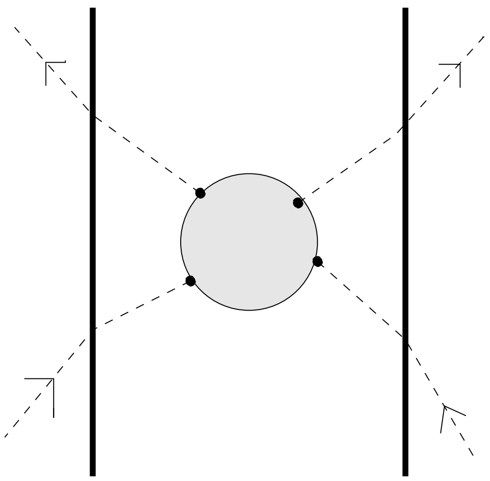

Figure 4: Scattering from an AdS bubble; here the dark lines denote a finite distance boundary.

3.1 S-Matrix: Poincaré coordinates

In Poincaré coordinates for AdS_5, a convenient example of a bubble geometry is provided by the asymptotically flat 3-brane. This spacetime has an asymptotically flat region that patches onto an AdS throat. The duality of the bubble to a CFT accounts for the success of CFT computations of supergravity scattering from 3-branes. This interpretation has been extensively explored in the work on brane and black hole greybody factors (e.g. [22]) and led directly to the formulation of the AdS/CFT correspondence in [2]. So we will only review the outlines of the argument here, and a pose a puzzle regarding modes with spacelike momenta.

The asymptotically flat, extremal 3-brane has a metric

$$ds^2 = H^{-1/2}\,(-dt^2 + d\vec{x}^2) + H^{1/2}\,(dr^2 + r^2\, d\Omega_5^2)\,, \qquad (3.2)$$

with $H = 1 + R^4/r^4$, $R^4 = 4\pi g\alpha'^2 N$ and a horizon at $r = 0$. When $r \ll R$ we can set $H \approx R^4/r^4$. Then taking $z = R^2/r$ as the new radial coordinate yields Poincaré AdS_5. Alternatively, when $r \gg R$, $H \approx 1$ giving flat space.

Consider a massless scalar ϕ satisfying the free wave equation in this background. Supergravity calculations of the scattering of this scalar from the AdS region will be valid when the 3-brane scale is large and incident energy small compared to the string scale. This was formalized in a "double scaling limit" as [9, 10]:

$$R^4/\alpha'^2 \sim gN \to \infty\,, \qquad \omega^2\alpha' \to 0\,. \tag{3.3}$$

Then setting $\phi = e^{i\vec{k}\cdot\vec{x}-i\omega t}\, Y_{lm}(\Omega)\,\psi$ gives:

$$\left[\left(1+\frac{R^4}{r^4}\right)(\omega^2-\vec{k}^2) - \frac{l(l+4)}{r^2} + \frac{1}{r^5}\partial_r r^5 \partial_r\right]\psi = 0\,. \tag{3.4}$$

Euclidean AdS enjoys the same wave equation with ω^2 replaced by $-\omega^2$.

Timelike momenta. When $q^2 = \omega^2 - \vec{k}^2 > 0$ solutions in the near and far region are Bessel functions. We want a scattering solution that is pure infalling at the brane horizon at $r = 0$ (or $z = \infty$ in the inverted AdS coordinate) and is smooth in the far region. So we choose:

$$\begin{aligned} \text{near}: &\quad \psi \propto z^2\, H^{(1)}_{l+2}(qz)\,, \\ \text{far}: &\quad \psi \propto r^{-2}\, J_{l+2}(qr)\,, \end{aligned} \tag{3.5}$$

with $z = R^2/r$ and match these solutions at $r \approx R$.[8] The bulk-boundary propagator in (2.14) now appears as the infalling disturbance created by a probe from the asymptotic region. The CFT correlators in (3.1) then describe the non-trivial interaction of these disturbances in the AdS region. Pure ingoing boundary conditions are also required for a smooth euclidean continuation — $H^{(1)}_\nu$ continues to K_ν which vanishes exponentially at the horizon while the other solution diverges. This is tantamount to choosing Hartle-Hawking boundary conditions for the black 3-brane. Different choices of bulk-boundary propagator for AdS/CFT would correspond in the AdS bubble interpretation to different boundary conditions for the 3-brane.

Spacelike momenta. The propagator (2.14) is defined even for spacelike momenta with $q^2 = \omega^2 - \vec{k}^2 < 0$. In this case, setting $q = \sqrt{|q^2|}$, and demanding regularity at the horizon and at infinity gives the near and far solutions:

$$\begin{aligned} \text{near}: &\quad \psi \propto z^2\, K_{l+2}(qz)\,, \\ \text{far}: &\quad \psi \propto r^{-2}\, K_{l+2}(qr)\,. \end{aligned} \tag{3.6}$$

The solutions vanish exponentially both at the horizon ($z = \infty$) and at infinity ($r = \infty$), and so are not scattering states. The interpretation of (3.1) for spacelike momenta therefore remains puzzling.

[8]In fact, the wave equation can be solved exactly in the 3-brane background in terms of Mathieu functions [23].

3.2 S-Matrix: global coordinates

In order to interpret (3.1) in global coordinates we want to construct a CAdS bubble inside a spacetime that admits propagating asymptotic states in a single asymptotic region. A stable string solution of this kind is not known.[9] Nevertheless, for purposes of illustration, consider a spacetime with the metric:[10]

$$\begin{aligned} r < r_b: \qquad & ds^2 = -(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2\,, \\ r > r_b: \qquad & ds^2 = -(1+r_b^2)d\tau^2 + \frac{dr^2}{1+r_b^2} + r^2 d\Omega^2\,. \end{aligned} \tag{3.7}$$

The interior metric is CAdS in coordinates $\sec^2\rho = 1 + r^2$ and we have set the AdS scale R to 1. The induced metrics match on the surface $r = r_b$. This metric is not expected to be a solution to string theory, and initial data corresponding to such a geometry would certainly evolve with time. Nevertheless, *any* string solution containing a metastable AdS bubble will share certain features of scalar field propagation that we now study.

We want to compute the effective potential seen by a massless scalar of angular momentum l propagating in (3.7). It is useful to work in terms of the tortoise coordinate r_* satisfying $dr_*/dr = \sqrt{-g_{rr}/g_{\tau\tau}}$. Let us expand the scalar ϕ in spherical harmonics as $\phi = Y_{l\vec{m}}\, u/R^{3/2}$ where R^2 is the coefficient of $d\Omega^2$ in (3.7). Then radial motion is governed by the action:

$$S_{eff} \propto \int d\tau\, dr_* \left[(\partial_\tau u)^2 - (\partial_{r_*} u)^2 - V(r_*)u^2\right], \tag{3.8}$$

with effective potential

$$V(r_*) = -g_{\tau\tau}\frac{l(l+1)}{R^2} + \frac{\partial^2_{r_*} R^{3/2}}{R^{3/2}}\,. \tag{3.9}$$

For $r < r_b$, the effective potential is that of anti-de Sitter space, and for $r \gg r_b$

$$V \sim \frac{1}{{r_*}^2}\,, \tag{3.10}$$

allowing a definition of asymptotic states.

Consider an incident wave moving towards $r = 0$ from $r \gg r_b$. Within the AdS region we want a solution that is smooth at $r = 0$. This selects the mode (2.19). Matching to the exterior incident wave would give an absorption coefficient. We are interested in a somewhat different analysis where we construct a number of

[9] Of course the full analytic continuation of the 3-brane metric has a CAdS interior and an infinite number of asymptotic regions [24] whose interpretation can be problematic.

[10] This simple metric is *not* asymptotically flat, although it is possible to write more complicated ones that are. Nevertheless, it permits the definition of asymptotic particle states.

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well separated wavepackets in the asymptotic region and focus them to meet and interact within the AdS bubble. The actual interaction within AdS is then completely described by the CFT correlators in (3.1). To compare to the full S-matrix computed in the asymptotic region we need the probability of penetrating into AdS, which will provide the correct normalization of CFT fields. The latter data depend on the precise patching of the AdS bubble into the asymptotic region. The CFT correlators in (3.1) summarize the universal, and nontrivial, part of the scattering amplitude which we can isolate by taking $r_b \to \infty$ in (3.7). In this limit, we recover CAdS. So, although the modes (2.19) are not normalizable and fluctuating within the full CAdS, the expression (3.1) is a universal limiting S-matrix for scattering from a large CAdS bubble.

This S-matrix interpretation also explains the origin of the singularities of the bulk-boundary propagator at the magic frequencies (2.22). As discussed in sec. 2, there is a spectrum of normalizable states in AdS at the frequencies $\omega_{nl} = 2h_+ + l + n$. While a wave incident on a CAdS bubble at generic frequencies will mostly reflect, near resonance ($\omega \approx \omega_{nl}$) the wavefunction for $r < r_b$ will scale as $1/(\omega - \omega_{nl} - i\Gamma)$. Here Γ is a decay width for states inside the bubble and $\Gamma \to 0$ as $r_b \to \infty$. The peak in the scattering amplitude at resonance becomes a pole as the CAdS states become stable in the $r_b \to \infty$ limit. This explains the origin of the otherwise disturbing singularity in the bulk-boundary propagator at the magic frequencies.

3.3 Interpretation

We should stress that the bubble construction used to interpret (3.1) is simply a useful artifice that can be removed. The non-trivial physics summarized in (3.1) is precisely that of scattering inside AdS. An alternate way to see this is to consider introducing sources for particles at finite but large radius in AdS. In the limit as the source goes to the boundary, and at the same time has its amplitude scaled up to compensate for the vanishing tunneling factors, we also recover (3.1).

Furthermore, our discussion has used a diagrammatic expansion for the nearly free limit which is suspect in the fully holographic theory. Nonetheless, our interpretation of (3.1) as a natural analogue of an S-matrix can be expected to hold in the full theory, providing a path from boundary to bulk physics.

4. States and transition amplitudes

In the previous section, we have found an interpretation for truncated time-ordered bulk Green functions convolved against non-normalizable modes. We might expect that Green functions convolved against normalizable modes have a meaning as well. Indeed, the conventional LSZ prescription in flat space relates S-matrix elements, defined as the overlap of "in" and "out" states with well-defined particle number, to truncated Green functions convolved against one-particle wavefunctions. Therefore, in this section we return to the issue of fluctuating states in AdS and transition amplitudes between them. We will show in sec. 4.1 that this prescription fails in CAdS space because "in" and "out" states, propagating as a collection of widely separated single non-interacting particles, do not exist. Roughly this is because lightlike geodesics hit the boundary in finite time, while spacelike geodesics always reflect back into the bulk after finite time. So particles (classical or quantum) do not become infinitely separated in the far past. To avoid this subtlety we will partition AdS into patches within which geodesics do not reconverge, and show that the appropriate CFT dual computes transition amplitudes between fluctuating states defined on the patch.

4.1 The failure of LSZ in CAdS

To define "in" and "out" states in the usual manner we follow sec. 3.1 of [25]. Assume we can split up the hamiltonian as:

$$H = H_0 + H_I \,, \tag{4.1}$$

where H_0 is a free hamiltonian, but contains the renormalized masses so that H_0 and H have the same one-particle spectrum. Then in Heisenberg picture, both free and interacting one-particle states satisfy:

$$H_0 \Psi_{0,\alpha} = E_\alpha \Psi_{0,\alpha} \,, \qquad H \Psi_\alpha = E_\alpha \Psi_\alpha \,. \tag{4.2}$$

Here α denotes the full set of quantum numbers–momenta, spin, and so on. "In" states ψ^+_α are defined by the requirement that wavepackets constructed from them approach free wavepackets at early times:

$$e^{-iHt} \int d\alpha \, g(\alpha) \, \Psi^+_\alpha \overset{t\to-\infty}{\longrightarrow} e^{-iH_0 t} \int d\alpha \, g(\alpha) \, \Psi_{0,\alpha} \,. \tag{4.3}$$

Here $g(\alpha)$ is a kernel used to define wavepackets. "Out" fields Ψ^-_α are defined identically, with the limit replaced by $t \to \infty$.

The Lippmann-Schwinger equation provides a recursive solution:

$$\Psi^\pm_\alpha = \Psi_{0,\alpha} + \int d\beta \, \frac{\langle \Psi_{0,\beta} | H_I | \Psi^\pm_\alpha \rangle}{E_\alpha - E_\beta \pm i\epsilon} \, \Psi_{0,\beta} \,. \tag{4.4}$$

Now we multiply both sides by $g(\alpha) e^{-iE_\alpha t}$ and sum over α. Wave packets built from (4.4) satisfy (4.3) if:

$$\lim_{t\to\mp\infty} \int\int d\alpha \, d\beta \, e^{-iE_\alpha t} \, g(\alpha) \, \frac{\langle \Psi_{0,\beta} | H_I | \Psi^\pm_\alpha \rangle}{E_\alpha - E_\beta \pm i\epsilon} \, \Psi_{0,\beta} = 0 \,. \tag{4.5}$$

We will check this condition in Minkowski and CAdS spaces.

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Minkowski spacetime. In Minkowski space, the energy spectrum is continuous. Extending the integral in (4.5) over positive and negative energies and integrating over α, the contour is closed in the upper half-plane for "in" states. The pole in the denominator is then avoided. There may be additional poles in $g(\alpha)$ controlling the energy width of the state, and poles in the matrix element controlling the duration of collision. For large, negative t, the integral will be a sum over these poles and exponentially damped. "Out" states may be treated similarly

CAdS spacetime. In CAdS the energies of the particles are discrete and given by (2.22). This spectrum is essentially dictated by the representation theory of the AdS isometry group; so we do not expect the integral spacing of levels to be affected by quantum corrections. Then k particles with masses m_i have a total energy,[11]

$$E = \sum_{i=1}^{k} \left(2h_+^{(k)} + l_k + 2n_k \right) , \tag{4.6}$$

and the left side of (4.5) is almost periodic in time with period 2π, up to an overall phase

$$e^{-i\left(\sum_{i=1}^{k} 2h_+^{(k)}\right)t} .$$

Thus the last term in (4.4), summed against $g(\alpha)\, e^{-iE_\alpha t}$, is finite and oscillatory in time, and the conditions in (4.3) cannot be met. This almost-periodic behavior of wavefunctions is the quantum reflection of the classical statement that geodesics are periodic in global time.

4.2 Transitions between states

Despite the subtlety described above we might expect to construct physically interesting transition amplitudes between states within a patch of AdS in which geodesics are not periodic. A particularly convenient patch is provided by the Poincaré coordinates discussed in sec. 2. The CAdS and PAdS metrics were presented in sec. 2 and, as shown in fig. 3, the global spacetime can be constructed as a PAdS ladder. We will refer to the nth patch as PAdS_n. Each patch has a timelike boundary B_n and a past (H_n^-)and future (H_n^+) horizon.

CAdS from PAdS. To define the theory on PAdS_n we need boundary conditions at B_n, H_n^+ and H_n^-. Boundary data on B_n are specified by turning on frozen, non-normalizable solutions of the form [17, 8, 7]:

$$\psi_{\mathbf{k}}(\mathbf{x}) = A\, z^{d/2}\, H_\nu^{(1)}(qz)\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} . \tag{4.7}$$

[11] Generally, the multiparticle energy may not be a sum of single particle energies; this formula is for illustration's sake.

For timelike frequenices($q^2 = \omega^2 - \vec{k}^2 > 0$), these modes are pure ingoing (outgoing) at the horizon for $\omega > 0$ ($\omega < 0$),

$$\psi_{\mathbf{k}}(\mathbf{x}) \stackrel{z\to\infty}{\longrightarrow} B\, z^{(d-1)/2}\, e^{i(qz-\omega t)+i\vec{k}\cdot\vec{x}}\,. \tag{4.8}$$

Modes with spacelike frequencies ($q^2 = \omega^2 - \vec{k}^2 < 0$) vanish exponentially at the horizon and are not of interest to us here. The normalizable mode solutions of the wave equation are [14, 15, 8]:

$$\phi_{\mathbf{k}}(\mathbf{x}) = A\, z^{d/2}\, J_\nu(qz)\, e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}} \tag{4.9}$$

and are a mixture of outgoing and ingoing modes at the horizon

$$\phi^1_{\mathbf{k}}(\mathbf{x}) \stackrel{z\to\infty}{\longrightarrow} B\, z^{(d-1)/2}\, \left[e^{i(qz-\omega t-\frac{\pi\nu}{2}-\frac{\pi}{4})+i\vec{k}\cdot\vec{x}} + e^{-i(qz+\omega t-\frac{\pi\nu}{2}-\frac{\pi}{4})+i\vec{k}\cdot\vec{x}}\right]. \tag{4.10}$$

We can patch together these modes on a series of patches to obtain a solution to the CAdS wave equation. For example, suppose that the mode $\phi_{\mathbf{k}}$ is present on PAdS$_n$. To match the flux at the horizon we could turn on $\psi_{\mathbf{k}}$ with positive frequency on PAdS$_{n-1}$ and $\psi_{\mathbf{k}}$ with negative frequency on PAdS$_{n+1}$. A general solution on CAdS which vanishes at very early and very late times is constructed by turning on a collection of ψ with positive frequency on some early patch, matching onto a sequence of normalizable and non-normalizable modes in later patches, and then soaking up the flux by a collection of ψ with negative frequency on a late patch. This process of splicing solutions is displayed schematically in fig. 5.

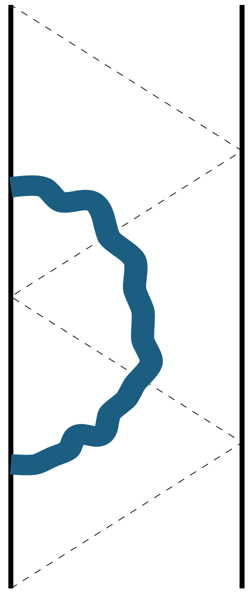


Figure 5: CAdS solutions from PAdS.

We have just described how to constructed classical solutions to the CAdS wave equations by sewing PAdS modes together. In the classical limit, given Dirichlet boundary conditions for PAdS$_n$, the normalizable modes (4.9) propagate undisturbed from the past to the future horizon. In the interacting theory, these modes are the candidate early and late time states between which we wish to compute transition amplitudes. In the following, we will describe how to do this from the dual CFT perspective.

CFT dual. The construction of CAdS as a tower of PAdS patches is dual to the definition of lorentzian CFTs on a stack of Minkowski diamonds [18]. The classical limit in spacetime is dual to the large N limit of the CFT, and the splicing prescriptions for classical solutions translate into relations between the CFTs defined on the boundaries B_n of the tower of patches. Let $\mathcal{O}_n$ be the CFT operator on B_n that is dual to a field ϕ. Deforming the large N CFT on B_{n-1} by adding the term $\int_\partial \phi_0(\mathbf{b})\, \mathcal{O}_{n-1}(\mathbf{b})$ to the lagrangean is dual in spacetime to a classical non-normalizable mode that approaches $\phi_0(\mathbf{b})$ on the boundary [2, 3, 8]. Working in a Fourier basis, a source for $\mathcal{O}_{n-1}(\mathbf{k})$ in the CFT is therefore dual to turning on the bulk-boundary propagator (2.14) (or $\psi_{\mathbf{k}}$) as a classical mode in the bulk. The patching conditions for CAdS then induce a classical mode in PAdS$_n$ to soak up the incoming flux at the past horizon. Choosing the normalizable mode $\phi_{\mathbf{k}}$ is dual to placing the CFT on B_n in the corresponding "coherent" state [7]. Finally, we can match onto a negative frequency $\psi_{\mathbf{k}}$ mode in PAdS$_{n+1}$. This naturally provides a source term for a negative frequency mode of $\mathcal{O}_{n+1}$.

From this we learn that states in PAdS$_n$ can be created (annihilated) by positive (negative) frequency operators $\mathcal{O}$ acting on past (future) patches. We are interested in transitions between states of PAdS$_n$. So it is sufficient to consider operators acting on B_{n-1} and B_{n+1}. Writing the positive and negative frequency parts of $\mathcal{O}$ as $\mathcal{O}^+$ and $\mathcal{O}^-$, incoming states $|s\rangle_n$ at H_n^- are defined as:

$$|s\rangle_n \equiv \mathcal{O}^+_{n-1}(\mathbf{k}_1)\cdots\mathcal{O}^+_{n-1}(\mathbf{k}_s)|0\rangle_{n-1}\,. \tag{4.11}$$

Here the incoming state at the past horizon of the nth patch has been identified with the action of positive frequency operators on the Poincaré vacuum state of B_{n-1}. Likewise, the outgoing state at the future horizon can be written as:

$${}_n\langle s'| \equiv {}_{n+1}\langle 0|\mathcal{O}^-_{n+1}(\mathbf{k}_s)\cdots\mathcal{O}^-_{n+1}(\mathbf{k}_1)\,. \tag{4.12}$$

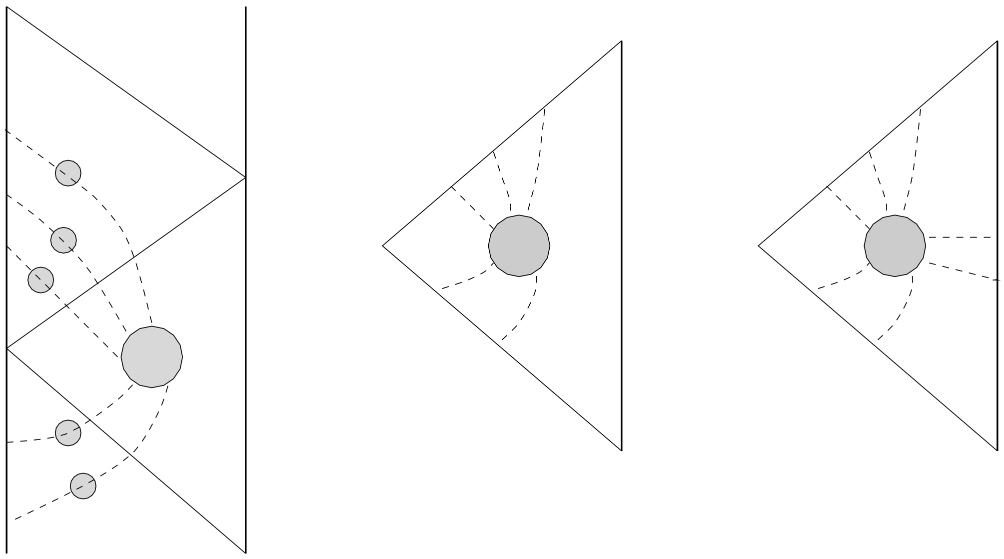
Here the action of negative frequency operators on the vacuum of B_{n+1} has been identified with the outgoing state on PAdS$_n$. In the nearly free limit, this associates the normalizable modes (4.9) with the in and out states of PAdS. But (4.11) and (4.12) have the added virtue of being well-defined even when interactions are turned on at finite N. What is more, these definitions accord well with our intuition that states in a CFT are created and annihilated by operators acting at early and late times. Here, operators on B_{n-1} and B_{n+1} are acting before and after the beginning and end of time from the PAdS$_n$ perspective.

PAdS transition amplitudes. We finally have all the ingredients to assemble transition amplitudes in PAdS$_n$ from the CFT perspective. Transition amplitudes in PAdS$_n$ are defined as the overlaps $\langle s'|s\rangle$. From the defintion of the in and out states, the calculation we must perform is:

$$\langle s'|s\rangle = \langle 0|\prod_i \mathcal{O}^-_{n+1}(\mathbf{k}_i)\prod_j \mathcal{O}^+_{n-1}(\mathbf{k}_j)|0\rangle\,. \tag{4.13}$$

The correlation functions are computed on the CAdS cylinder where the CFT is actually defined using the master formula (2.11). The notation $\mathcal{O}_m(\mathbf{k})$ indicates the Fourier mode of an operator which has support only on the patch B_m of the CAdS boundary cylinder. So, following (3.1), after Fourier transforming we compute:

$$\langle s'|s\rangle = \int \prod_{i=1}^{k}\prod_{j=1}^{l}\Big(d\mathbf{x}_i\,d\mathbf{y}_j\,\psi^-_{\mathbf{k}_i}(\mathbf{x}_i)\,\psi^+_{\mathbf{k}_j}(\mathbf{y}_j)\Big)\,\langle T\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_k)\phi(\mathbf{y}_1)\cdots\psi(\mathbf{y}_l)\rangle_{\mathrm{T}}\,. \tag{4.14}$$



(a) (b) (c)

Figure 6: PAdS diagrams computing transition amplitudes.

Here $\mathbf{x}$ and $\mathbf{y}$ are CAdS coordinates while $\psi^{\pm}$ are positive and negative frequency, non-normalizable Poincaré modes of the form (4.7) written in global coordinates. We then interpret (4.13) as a transition amplitude for states on PAdS_n. Diagramatically we obtain figs. $6a$ and $6b$ where the truncation of CAdS diagrams by the PAdS patch leaves legs intersecting the past and future horizons. These legs are wavefunctions representing the in and out states and, in the nearly free limit, will be given in PAdS_n by the normalizable AdS modes. The shaded circles on the propagators from the CAdS boundary to the PAdS horizons indicate that the full interacting propagator should be used to define the meaning of in and out states of PAdS_n. In general this means that these states do not have a clear interpretation as single particle wavefunctions, but they nevertheless give a basis for the initial and final time boundary conditions for PAdS.

In addition to the transition amplitudes (4.13) we may compute amplitudes like

$$\langle s'|\mathcal{O}_1(\mathbf{k}_1)\cdots\mathcal{O}_n(\mathbf{k}_m)|s\rangle\,. \tag{4.15}$$

Diagramatically these are represented in fig. $6c$. Following previous sections these amplitudes may be interpreted as amplitudes for scattering from an excited AdS bubble. The local operator insertions on the PAdS_n boundary appear in the bulk as non-normalizable modes while the states are represented by normalizable modes in the nearly-free limit.

4.3 Discussion

Conclusion and subtleties. At first glance (4.14) appears to describe PAdS transition amplitudes as truncated bulk Green functions convolved again normalizable wavefunctions. This interpretation basically applies in the nearly free limit, leading to fig. 6a. In general, however, the interactions in (4.14) may occur anywhere within CAdS, leading to fig. 7. One way of dealing with this subtlety is to create states via well separated wavepackets in PAdS$_{n-1}$ that do not interact until they have entered PAdS$_n$. This is certainly possible in the semiclassical limit, and in this way the in-states on the past horizon remain under reasonable control. Another approach is to simply define the states on the past (future) horizon to be the objects on those surfaces that result via propagation through the previous (later) patches. In the latter approach, as illustrated in fig. 7, the particle number of the in and out states may be indefinite; but these states are nonetheless well defined via the operator constructions (4.11) and (4.12).

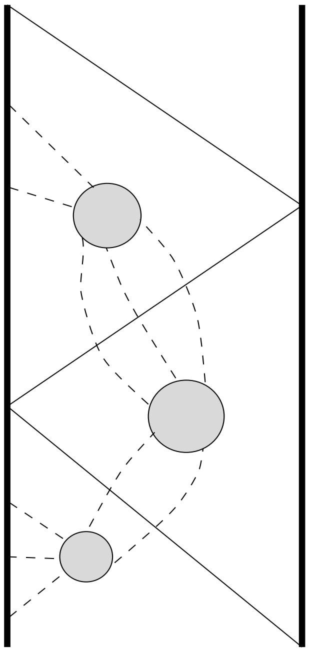


Figure 7: Contributions from the truncated Green function outside PAdS$_n$.

Truncated AdS. Instead of building up CAdS as a sequence of PAdS patches we could simply construct a sequence of cylinders of length T in global time. We will call the nth cylinder TAdS$_n$. As we have discussed, the effect of acting on the boundary with a CFT operator is transported into spacetime by the bulk-boundary propagator. By definition, this propagator in position space is a delta function on the boundary. It follows from this that a bulk-boundary propagator with support only on the TAdS$_{n-1}$ boundary must create a solution that is expandable in the normalizable mode solutions of the next patch TAdS$_n$. We can therefore precisely mimic the PAdS construction above to construct transition amplitudes for TAdS states. Again, in the nearly free limit, these are normalizable modes convolved against truncated Green functions, all in patch n, but a more complicated interpretation applies in the fully interacting theory.

Flat space S-matrix from AdS. The methods of this section were designed to describe the scattering of wavepackets created on the early and late boundaries of a patch. In the very large N limit, the spacetime is nearly flat in the interior. Our techniques can readily be used to to construct wavepackets within the flat region of the bulk spacetime. Then (4.13) gives a holographic computation of transition amplitudes between states in flat space. A very similar logic has been pursued recently in [11, 12] where equations like (3.1), (4.13) were also interpreted as transition amplitudes.

5. Towards bulk correlators

It has been suggested that the on-shell transition amplitudes recovered in the preceding sections contain all the information we can extract from a holographic theory [26]. Nonetheless, we know that approximate spacetime locality holds in nature and we would like to see how it is encoded in the CFT.

Eqs. (2.15), (2.6) imply a relation between bulk and boundary correlators [6]:

$$\langle \mathcal{O}_1(\mathbf{b}_1)\cdots\mathcal{O}_n(\mathbf{b}_n)\rangle = \prod_{k=1}^{n} \lim_{z_k\to 0} z_k^{-\Delta_{(k)}} \langle \phi(z_1,\mathbf{x}_1)\cdots\phi(z_n,\mathbf{x}_n)\rangle\,. \tag{5.1}$$

where $\Delta_{(k)} \equiv 2h_{+(k)}$ is the conformal dimension of $\mathcal{O}_k$. This is true for general correlators, suggesting the relation

$$\mathcal{O}(\mathbf{b}) = \lim_{z\to 0} z^{-\Delta}\phi(z,\mathbf{x})\,. \tag{5.2}$$

This is also implied in the $N \to \infty$ limit by eq. (15) of [7]. If an inverse of the map (5.2) exists, we could reconstruct off-shell bulk correlators from boundary data, and directly investigate spacetime locality. This section will explore attempts to construct such an inverse.

5.1 The free field map

In the free limit we can invert (5.2) by comparing the bulk and boundary mode expansions [6, 7]. The bulk expansions for a scalar field of mass m are fixed by the equations of motion and canonical commutation relations in spacetime.[12] On the boundary we simply expand in Fourier modes and fix the normalization via (5.2).

5.1.1 Mode expansions

Global coordinates. The bulk mode expansion is:

$$\begin{aligned}\Phi(t,\rho,\Omega) = \sum_{n,\ell,\vec{m}} &\left(\frac{\Gamma(1+n)\Gamma(\Delta+\ell+n)}{\Gamma(\ell+\frac{d}{2}+n)\Gamma(1+\nu+n)N_\ell}\right)^{\frac{1}{2}} \times \\ &\times (\sin\rho)^\ell\,(\cos\rho)^\Delta\, P_n^{(\ell+\frac{d}{2}-1,\nu)}(\cos 2\rho) \times \left\{e^{-i\omega_{n\ell}t}\,Y_{\ell\vec{m}}(\Omega)\,\hat{a}_{n\ell\vec{m}} + \text{h.c.}\right\}.\end{aligned} \tag{5.3}$$

(See [15, 14, 16, 8] for a discussion of mode solutions.) The frequencies $\omega_{n\ell}$ are defined in eq. (2.22) and the spherical harmonics $Y_{\ell\vec{m}}$ satisfy the standard orthonormality conditions

$$\int d\Omega\, Y^*_{\ell\vec{m}}(\Omega)\, Y_{\ell'\vec{m}'}(\Omega) = \delta_{\ell\ell'}\,\delta_{\vec{m}\vec{m}'}\,N_\ell\,. \tag{5.4}$$

Then the creation and annihilation operators obey commutation relations

$$\left[\hat{a}_{n\ell\vec{m}},\, \hat{a}^\dagger_{n'\ell'\vec{m}'}\right] = \delta_{nn'}\,\delta_{\ell\ell'}\,\delta_{\vec{m}\vec{m}'}\,. \tag{5.5}$$

[12]The mass is assumed to include any curvature couplings.

The boundary expansion follows from Fourier expansion of the CFT operators and from (5.2):

$$\mathcal{O}(t,\Omega) = \frac{1}{\Gamma(1+\nu)} \sum_{n,\ell,\vec{m}} \left(\frac{\Gamma(1+n+\nu)\Gamma(\Delta+\ell+n)}{N_\ell \Gamma(1+n)\Gamma(\ell+\frac{d}{2}+n)} \right)^{\frac{1}{2}} \left\{ e^{-i\omega_{n\ell}t} Y_{\ell\vec{m}}(\Omega) \hat{a}_{n\ell\vec{m}} + \text{h.c.} \right\} \tag{5.6}$$

Poincaré coordinates. In PAdS $\Phi(z,b)$ can be expanded as

$$\Phi(z,t,\vec{x}) = \int d^{d-1}\vec{k}\, dq\, z^{\frac{d}{2}}\, J_\nu(qz) \left[\frac{q}{2(2\pi)^{d-1}\omega(q,k)} \right]^{\frac{1}{2}} \left\{ e^{-i\omega(q,k)t + i\vec{k}\cdot\vec{x}} \hat{b}_{q\vec{k}} + \text{h.c.} \right\}. \tag{5.7}$$

Here $\omega(q,k) = q^2 - k^2$. J_ν are Bessel functions, and we follow the conventions of [21, 20]. The creation and annihilation operators $(\hat{b}, \hat{b}^\dagger)$ satisfy the commutation relations

$$[\hat{b}_{q\vec{k}}, \hat{b}_{q'\vec{k}'}] = \delta(q-q')\delta^{(d-1)}(\vec{k}-\vec{k}'). \tag{5.8}$$

Fourier expanding the dual operator and using (5.2) gives:[13]

$$\mathcal{O}(t,\vec{x}) = \int dq\, d^{(d-1)}\vec{k} \left(\frac{q}{2}\right)^{\nu+\frac{1}{2}} \frac{1}{\Gamma(1+\nu)\left[(2\pi)^{d-1}\omega(q,k)\right]^{\frac{1}{2}}} \left\{ e^{-i\omega(q,k)t + i\vec{k}\cdot\vec{x}} \hat{b}_{q,\vec{k}} + \text{h.c.} \right\}. \tag{5.9}$$

5.1.2 Bulk operators from boundary operators?

The bulk and boundary mode expansions can be conveniently related via a "transfer matrix" [6]:

$$\phi(z,\mathbf{b}) = \int d\mathbf{b}'\, M(z,\mathbf{b};\mathbf{b}')\, \mathcal{O}(\mathbf{b}'). \tag{5.10}$$

The explicit transfer matrices are readily found by Fourier transforming eqs. (5.6) and (5.9). In CAdS this gives

$$\begin{aligned} M(\rho_1,t_1,\Omega_1;t_2,\Omega_2) = \sum_{n,\ell,\vec{m}} &\frac{\Gamma(1+\nu)\Gamma(1+n)}{N_\ell\Gamma(1+n+\nu)} (\sin\rho_1)^\ell (\cos\rho_1)^\Delta P_n^{(\ell+\frac{d}{2}-1,\nu)}(\cos 2\rho_1) \times \\ &\times \left\{ e^{-i\omega(t_2-t_1)} Y^*_{\ell\vec{m}}(\Omega_1) Y_{\ell\vec{m}}(\Omega_2) + e^{i\omega(t_1-t_2)} Y_{\ell\vec{m}}(\Omega_1) Y^*_{\ell\vec{m}}(\Omega_2) \right\}, \end{aligned} \tag{5.11}$$

and in Poincaré coordinates,

$$\begin{aligned} M(z_1,t_1,\vec{x}_1;t_2,\vec{x}_2) = \Gamma(1+\nu) \int d^{d-1}\vec{k}\, dq \left(\frac{2}{q}\right)^\nu z_1^{\frac{d}{2}}\, J_\nu(qz_1)\, \times \\ \times \left\{ e^{-i\omega(q)(t_2-t_1) + i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)} + e^{i\omega(q)(t_2-t_1) - i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)} \right\}. \end{aligned} \tag{5.12}$$

[13] The $z \to 0$ limit is formal and is taken term by term in the expansion in q; otherwise there is an order-of-limits issue in using the small-argument asymptotics for J_ν at the upper end of the integral.

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This gives a concrete proposal for the map from boundary to bulk fields, and, as in [6], we might now attempt to promote it to the interacting theory and infer local bulk correlators via

$$\langle\phi(z_1,\mathbf{b}_1)\phi(z_2,\mathbf{b}_2)\cdots\phi(z_n,\mathbf{b}_n)\rangle \stackrel{?}{=} \int\prod_{i=1}^{n}\left[d\mathbf{b}_i' M(z_i,\mathbf{b}_i;\mathbf{b}_i')\right]\langle\mathcal{O}(\mathbf{b}_1')\mathcal{O}(\mathbf{b}_2')\cdots\mathcal{O}(\mathbf{b}_n')\rangle\,. \tag{5.13}$$

However, as the authors of [6] noted, this cannot be correct because the result satisfies the free wave equation rather than the interacting Dyson-Schwinger equations. The next subsection explains this using the Lehmann representation for bulk fields.

In fact, from (3.1) we know that the right hand side of (5.13) can be expressed as:

$$\begin{aligned}\int\prod_{i=1}^{n}[\ d\ \mathbf{b}_i' M(z_i,\mathbf{b}_i;\mathbf{b}_i')]\langle\mathcal{O}(\mathbf{b}_1')\mathcal{O}(\mathbf{b}_2')\cdots\mathcal{O}(\mathbf{b}_n')\rangle =\\ = \int\left[\prod_{k=1}^{n} d\mathbf{x}_k'\right]\prod_{k=1}^{n}\left[d\alpha_k\,\phi^*_{\alpha_k}(\mathbf{x}_k)\,\psi_{\alpha_k}(\mathbf{x}_k')\right]\langle\phi_1(\mathbf{x}_1')\cdots\phi_n(\mathbf{x}_n')\rangle_T\,,\end{aligned} \tag{5.14}$$

where ϕ^*_α denotes a normalizable wavefunction and the integrals over α_k are shorthand for the full sum/integral over Fourier conjugate variables appropriate to either the global or Poincaré case. Prior to integrating over α_k, the right side of (5.14) is a truncated bulk Green function projected onto non-normalizable, on-shell wavefunctions. This is further convolved against normalizable modes in (5.14). So it is manifest that the right side of (5.13) satisfies a free wave equation.

5.2 A Lehmann representation for CAdS

Düsedau and Freedman [13] constructed a Lehmann representation for AdS using the decomposition of the field theory Hilbert space on AdS_{d+1} into representations of $\mathrm{SO}(d,2)$. The starting point is the identity written as a sum over conformal representations. Labeling the conformal weight by Δ and the elements of the representation by (n,l), we have:

$$\mathbb{1} = \int d\Delta \sum_{n,\ell,m}|\Delta;n,\ell,m\rangle\langle\Delta;n,\ell,m|\,, \tag{5.15}$$

where the energy of $|\Delta;n,\ell,m\rangle$ with respect to global time is $\omega^\Delta_{n\ell} = \Delta+\ell+2n$. In a Lehmann representation, the two point function of the full (interacting) scalar fields Φ can be written in terms of the two-point function of free scalar fields ϕ in representation Δ, integrated over Δ with a spectral weight $\rho(\Delta)$:

$$\langle T\left(\Phi(x)\Phi(y)\right)\rangle = \int d\Delta\,\rho(\Delta)\,G_F(\Delta;x-y)\,. \tag{5.16}$$

Here G_F is the Feynman propagator for a free scalar field with mass $m^2_\Delta = \Delta(\Delta-d)$.

Determination of ρ proceeds as in flat space. Consider a a canonically normalized free field of conformal weight Δ. It follows trivially that

$$\langle 0|\phi(x)|\Delta; n, \ell, m\rangle = \Phi^{+}_{n\ell m}(x)\,, \tag{5.17}$$

where Φ^+ is the normalizable wavefunction multiplying $\hat{a}_{n\ell m}$ in the expansion for ϕ. A general interacting scalar field Φ must have the same matrix element up to a normalization factor:

$$\langle 0|\Phi(x)|\Delta; n, \ell, m\rangle = N(\Delta)\Phi^{+}_{n\ell m}(x)\,, \tag{5.18}$$

for any Δ. Eq. (5.16), with $\rho(\Delta) = |N(\Delta)|^2$, then follows from inserting (5.15) into the right hand side of the two-point function.

A similar decomposition holds on the boundary, since its Hilbert space and symmetries are the same. The coordinate dependence in

$$\langle 0|\mathcal{O}(b)|\Delta; n, \ell, m\rangle = \hat{N}(\Delta)e^{-i\omega^{\Delta}_{n,\ell,m}t}Y_{\ell}(\Omega) \tag{5.19}$$

is again determined by symmetries. For a conformal primary operator $\mathcal{O}$ with dimension $\Delta(\mathcal{O})$, $\hat{N}$ will have support only at $\Delta = \Delta(\mathcal{O})$. This is compatible with the more complicated spectral decomposition of the dual operator Φ, so long as the minimal value of Δ for which $N(\Delta)$ has support is $\Delta(\mathcal{O})$. Eq. (5.1) implies that

$$\langle 0|\mathcal{O}(\Omega, t)|\Delta; \ell, n\rangle = \lim_{\rho\to\pi/2}(\cos\rho)^{-\Delta(\mathcal{O})}\langle 0|\Phi(\rho, \Omega, t)|\Delta; n, \ell, m\rangle\,. \tag{5.20}$$

The interacting field Φ may contain components of many different conformal weights, but the matrix element $\langle 0|\Phi(\rho, \Omega, t)|\Delta; \ell, n\rangle$ is a normalizable mode with fixed conformal weight Δ. So it falls off at the boundary as $\cos^{\Delta}\rho$, and the right side of (5.20) vanishes for any $\Delta > \Delta(\mathcal{O})$.

Eq. (5.20) shows that restricting Φ to the boundary isolates the representation of $\mathrm{SO}(2,d)$ with the lowest mass/conformal dimension in the interacting field. This should be thought of as a "single-particle state" corresponding to Φ, following the flat space analogy. The CFT primary $\mathcal{O}$ carries precisely this conformal dimension. The CFT operators function similarly to the "in" and "out" states:

$$\langle 0|\Phi|\Delta; n, \ell, m\rangle \overset{\rho\to\pi/2}{\longrightarrow} (\cos\rho)^{\Delta(\mathcal{O})}\langle 0|\mathcal{O}|\Delta; n, \ell, m\rangle \tag{5.21}$$

is the analog of the weak asymptotic condition

$$\langle 0|\Phi|\Delta; n, \ell, m\rangle \overset{t\to-\infty}{\longrightarrow} Z^{-\frac{1}{2}}\langle 0|\Phi_{\rm in}|\Delta; n, \ell, m\rangle \tag{5.22}$$

in flat-space field theory (cf. for example [27]).

Moral of the story. The fact that restricting the bulk interacting fields to the boundary isolates the single particle states teaches us two very important things. First we understand why (5.13) fails as an attempt to reconstruct local bulk correlators; the formula does not contain the "multi-particle part" of the bulk fields. Secondly, we understand why the S-matrix prescriptions of sec. 3 and sec. 4 work even in the interacting theory; boundary operators really do create acceptable single-particle asymptotic states. This feature of the map (5.2) gives us one way to think about the general problem of reconstructing interacting bulk fields from the boundary. Presumably, in order to create the "multi-particle" pieces of a bulk field we will have to write a CFT operator containing a sum of various products of CFT primaries and descendants.

5.3 Towards interacting fields

In this section we will try to construct the interacting field ϕ from the CFT. We will operationally reproduce a spacetime perturbation expansion via calculations in the large N CFT, where there is a free-field representation.

5.3.1 Spectral decomposition and propagators

Our starting points are the decompositions (5.6) and (5.9):

$$\mathcal{O}(t,\Omega) = \sum_{n,l} \mathcal{O}(n,\ell;t,\Omega) \tag{5.23}$$

and

$$\mathcal{O}(t,\vec{x}) = \int dq\, \mathcal{O}(q;\vec{x},t)\,, \tag{5.24}$$

which write the CFT operators as sums of appropriate Fourier modes. $\mathcal{O}(n,\ell)$ and $\mathcal{O}(q)$ may be extracted from $\mathcal{O}$ via a (nonlocal) transform following from (5.6) and (5.9). In the Poincaré case, (5.24) is an effective decomposition of $\mathcal{O}$ onto "mass-shells" of fixed $q^2 = \omega^2 - k^2$. Indeed, we can use (5.9) to suggestively write the two-point function as:

$$\begin{aligned}\Big\langle T\left(\mathcal{O}(t,\vec{x})\,\mathcal{O}(t',\vec{x}')\right)\Big\rangle &= \int dq\,\Big\langle T\left(\mathcal{O}(q;t,\vec{x})\,\mathcal{O}(q;t',\vec{x}')\right)\Big\rangle \\ &= \frac{1}{2^{2\nu+1}(\Gamma(1+\nu))^2}\int dq\, q^{2\nu+1}\,\Delta(q^2;t-t',\vec{x}-\vec{x}')\,,\end{aligned} \tag{5.25}$$

where $\Delta(q^2)$ is the Feynman propagator for a free particle with mass q^2 on $\mathbb{R}^{1,d-1}$. This is the spectral representation for the two-point function of a conformal primary with dimension $2h_+ = (d/2+\nu)$ (cf. [28]). A similar representation should exist for global coordinates.

The "transfer matrix" relation in sec. 5.1.2 between bulk and boundary fields was nonlocal in the boundary coordinates. In contrast, the large N "spectral fields"

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$\mathcal{O}(n,\ell)$ and $\mathcal{O}(q)$ are related to bulk free fields by *local* transformations:

$$\begin{aligned}\Phi(t,\rho,\Omega) &= \Gamma(1+\nu)\sum_{n,\ell,m}\left(\frac{\Gamma(1+n)}{\Gamma(1+n+\nu)}\right)^{\frac{1}{2}}\times \\ &\quad \times(\sin\rho)^{\ell}\,(\cos\rho)^{\Delta}\,P_n^{(\ell+\frac{d}{2}-1,\nu)}(\cos 2\rho)\,\mathcal{O}(n,\ell;t,\Omega) \\ &\equiv \sum_{n,\ell}\alpha(n,\ell;\rho)\,\mathcal{O}(n,\ell;t,\Omega) \end{aligned} \tag{5.26}$$

and

$$\Phi(z,t,\vec{x}) = \Gamma(1+\nu)\int dq\,\left(\frac{2}{q}\right)^{\nu} z^{\frac{d}{2}}\,J_\nu(qz)\,\mathcal{O}(q;t,\vec{x}) \equiv \int dq\,\beta(q;z)\,\mathcal{O}(q;t,\vec{x})\,. \tag{5.27}$$

The latter formula implies that free bulk operators arise from a Bessel transformation of large N spectral fields on the boundary.

The simplicity of these relations extends to bulk two-point functions. If we attempt to write these using (5.10), the time-ordering operator fails to push through to the boundary fields and a cumbersome expression results. The spectral fields yield a simpler result:[14]

$$\begin{aligned}\langle T\left(\Phi(z_1,\mathbf{b}_1)\Phi(z_2,\mathbf{b}_2)\right)\rangle &= \int dq_1 dq_2 \beta(q_1;z_1)\beta(q_2;z_2)\langle T\left(\mathcal{O}(q_1;\mathbf{b}_1)\mathcal{O}(q_2;\mathbf{b}_2)\right)\rangle \qquad (5.28)\\ &= \frac{1}{2^{2\nu+1}(\Gamma(1+\nu))^2}\int dq q^{2\nu+1}\beta(q;z_1)\beta(q;z_2)\Delta(q^2;\mathbf{b}_1-\mathbf{b}_2)\,.\end{aligned}$$

Here the only transformation we have done is between bulk radial coordinate and CFT mass-shell, and the time-ordering remains untouched.

5.3.2 Interactions

Equation (5.2) appears to relate the bulk interacting fields to the boundary operators. However, we showed that it does so in a way that projects onto the non-interacting part of the field. Is there a prescription to recover the full bulk fields, or equivalently their correlators?

There is no reason to expect that the interacting fields can be linearly expanded in terms of creation and annihilation operators obeying (5.5) or (5.8). Instead, they should contain products of the free field creation and annihilation operators, corresponding to "multi-particle" interacting states. At least in the semiclassical (large N) limit, this distinction is captured in the relation between Heisenberg and interaction picture operators:

$$\Phi_H(x,t) = e^{-iHt}e^{iH_0t}\Phi_I(x,t)e^{-iH_0t}e^{iHt}\,. \tag{5.29}$$

Their correlators are related by

$$\langle T\left(\phi_{1,H}(z_1,\mathbf{b}_1)\ldots\phi_{n,H}(z_n,\mathbf{b}_n)\right)\rangle = \langle T\left(\phi_{1,I}(z_1,\mathbf{b}_1)\ldots\phi_{n,I}(z_n,\mathbf{b}_n)e^{-i\int dt\mathcal{H}_{\rm int}(\mathbf{b},z,t)}\right)\rangle\,, \tag{5.30}$$

[14]For simplicity we focus on the Poincaré case.

where $\mathcal{H}_{\rm int}$ is the interaction hamiltonian. Within perturbative supergravity, $\mathcal{H}_{\rm int}$ is well-defined and the interaction terms are suppressed by powers of $1/N$ [1].

We would like to reproduce this perturbative picture from the boundary perspective. In the finite N CFT it is far from clear that there is a corresponding definition of the "interaction representation". However, just as (5.30) is an expansion around the free limit, we can attempt a construction about the infinite N CFT. The idea is to begin with the free-field operators, construct the bulk interaction picture fields via eqs. (5.10), (5.26), or (5.27), and then infer from the bulk lagrangean the corresponding perturbation expansion in the CFT.

This procedure is most transparent in the language of the spectral fields. Suppose that the interaction hamiltonian is a simple coupling of three scalars ϕ_i,[15]

$$H_{\rm int} = \frac{\lambda}{N}\int dz\, d^{d-1}\vec{x}\, \phi_i(z,\mathbf{b})\,\phi_j(z,\mathbf{b})\,\phi_k(z,\mathbf{b})\,. \tag{5.31}$$

We transfer this to the boundary using the Bessel transformation in (5.27), and integrate over z. This gives:

$$H_{\rm int}(t)\left(\{\mathcal{O}(q)\}\right) = \int dq_i\, dq_j\, dq_k\, d^{d-1}\vec{x}\, V(q_i,q_j,q_k)\, \mathcal{O}_i(q_i;t,\vec{x})\, \mathcal{O}_j(q_j;t,\vec{x})\, \mathcal{O}_k(q_k;t,\vec{x})\,, \tag{5.32}$$

where

$$V(q_i,q_j,q_k) = \frac{\lambda}{N}\int dz\, \beta(q_i;z)\,\beta(q_j;z)\,\beta(q_k;z) \tag{5.33}$$

and $\mathcal{O}_i(\mathbf{b})$ are the boundary operators dual to the scalars. Other terms in the bulk interaction hamiltonian can also be converted into functionals of $\mathcal{O}(q)$ in the obvious way. The bulk correlator (5.30) is then:

$$\begin{aligned}\langle T\left(\phi_{1,H}(z_1,\mathbf{b}_1)\ldots\phi_{n,H}(z_n,\mathbf{b}_n)\right)\rangle = \int \prod_{i=1}^{n} dq_i\, \beta(q_i;z_i)\, \times \\ \times\langle T\left(\mathcal{O}(q_1;\mathbf{b}_1)\ldots\mathcal{O}(q_n;\mathbf{b}_n)e^{-i\int dt H_{\rm int}(\{\mathcal{O}(q)\})}\right)\rangle\,.\end{aligned} \tag{5.34}$$

The expectation value in the second line can be computed perturbatively in $H_{\rm int}$ using Wick's theorem and the propagator in eq. (5.26). This propagator can in turn be derived by treating $\mathcal{O}(q)$ as a field with an appropriate lagrangean. Including interaction terms, we can therefore imagine deriving the amplitudes perturbatively from

$$\begin{aligned}\mathcal{L} = \sum_k \int dq_k\, d\mathbf{b}\, \frac{2^{2\nu_k+1}\Gamma(1+\nu_k)^2}{q_k^{2\nu_k+1}} \left[(\partial\mathcal{O}_k(q_k;\mathbf{b}))^2 + q_k^2\mathcal{O}_k(q_k;\mathbf{b})^2\right] + \\ +\frac{1}{N}\sum_{ijk}\int d\mathbf{b}\, dq_i\, dq_j\, dq_k\, V(q_i,q_j,q_k)\, \mathcal{O}_i(q_i;\mathbf{b})\, \mathcal{O}_j(q_j;\mathbf{b})\, \mathcal{O}_k(q_k;\mathbf{b}) + \cdots\end{aligned} \tag{5.35}$$

[15] Again, we focus on the Poincaré case.

This should be interpreted as a lagrangean for the "collective fields" $\mathcal{O}(q)$, and allows us to compute CFT correlators in a power series in $1/N$.[16]

The dimension of $\mathcal{O}$ is encoded in the normalization factor

$$A(q,\nu) = \frac{2^{2\nu+1}\Gamma(1+\nu)^2}{q^{2\nu+1}}, \tag{5.36}$$

and the operator product coefficients for the primary operators are encoded in the three-point couplings $V(q_1, q_2, q_3)$.

5.4 Interpretation

Our collective field lagrangean (5.35) is local in spacetime after transforming q to z via the kernel β, and completely captures the leading perturbative supergravity. It is true that we have constructed this theory from supergravity by fiat, but we have thus found a fairly simple but not *a priori* obvious integral transformation between the resulting collective fields and the $N = \infty$ CFT variables. We can now pose sharp questions directly within the CFT about the reconstruction of local spacetime physics. Can we reproduce a collective field lagrangean like (5.35) (and therefore bulk supergravity) directly from the gauge theory via standard large N techniques? Can such a representation of the theory be found at finite N? In particular, the loop equations for 4d $N = 4$ Yang-Mills theory, which we might hope reproduces string dynamics on $\mathrm{AdS}_5 \times S^5$, should contain our collective field theory in some $\alpha' \to 0$ approximation. Furthermore, the integral transformation between bulk and boundary fields (5.27) is strongly reminiscent of the relation between correlators of collective fields and tachyon fields in the $c = 1$ matrix model of 2D string theory [30]. We suspect that this similarity is not accidental.[17]

It would be interesting to see how the discussions in [6, 7] relate to this manifestly local description. In those papers, it was found that for classical probes, bulk locality in z translated to a sort of "scale-size locality" in the large-N boundary theory. In eq. (5.27), q functions as a scale, selecting a mass shell in the Fourier expansion of the primary operator $\mathcal{O}$. This is suggestive, but does not give a precise translation of bulk radial locality into CFT scales because q is related to z by an integral transform.

It is also amusing to note that the steps in section 5.3 can be reproduced in other spaces besides AdS, and used to define a "boundary theory" from a given bulk theory. One simple example defines a boundary lagrangean on the $\mathbb{R}^{d-1}$ boundary of the upper half space of $\mathbb{R}^d$. This observation may be relevant to exploration of holography in more general contexts.

[16] We are using the term "collective field theory" somewhat loosely. The feature our theory shares with collective field theory in the usual sense [29] is that we take an overcomplete basis of operators and promote them to independent lagrangean fields. And, as in [29], this lagrangean gives a $1/N$ expansion for exact correlators of the theory. It is clearly desirable to relate these two theories.

[17] In fact the authors of [31] have suggested that the $c = 1$ matrix model is a holographic description of 1+1-dimensional string theory, precisely in the fashion of the AdS/CFT correspondence.

Of course (5.35) cannot be the whole story. The expansion it provides in $1/N$ and $g^2_{\rm YM}$ will undoubtedly be asymptotic at best, and the Hilbert space of this collective field theory is not the same as the Hilbert space of the finite-N CFT.[18] However, this statement is precisely what makes the collective field theory (5.35) an interesting starting point for studying bulk dynamics. Classical gravity and its naïve semiclassical extension are afflicted with well-known pathologies. The degree to which our collective field theory is a poor approximation to the exact CFT should be precisely the degree to which our assumptions about classical and semiclassical gravity are invalid; the full CFT, which is manifestly well-defined, will remove the pathologies. How it does so can in principle be learned by understanding the difference between our collective field description and the full conformal field theory. This difference will contain the non-locality resulting from the holographic description.

6. Conclusion

In this article we have interpreted the correlation functions of the CFT duals of AdS spaces in terms of bulk S-matrices and transition amplitudes. First, vacuum correlators of the CFT were expressed in the bulk as truncated n-point functions convolved against non-normalizable on-shell modes. We interpreted these expressions as an S-matrix for AdS arising from a limit of scattering in asymptotically flat space from an AdS bubble. In the free limit, fields in AdS spacetime possess a class of normalizable, fluctuating solutions. A traditional LSZ prescription would compute transition amplitudes between these states. We showed that the usual LSZ framework fails in global AdS spacetime, essentially because the periodicity of geodesics obscures the definition of asymptotic states. We then avoided this subtlety by partitioning AdS into patches and showed that CFT correlation functions compute transition amplitudes between suitable states defined on these patches. These states correspond to boundary conditions on the early and late time surfaces of a patch and, unusually, were created and annihilated by the action of operators on the vacua of earlier and later patches.

Finally, we have tried to directly reconstruct local bulk operators from the boundary CFT. In the $N \to \infty$ limit, this reconstruction is straightforward. But at finite N, single CFT primaries capture only the "one-particle" parts of the dual bulk operators, making reconstruction a nontrivial, nonlinear process. We proposed a "collective field theory" which reproduces CFT correlators order by order in $g_{\rm YM}$ and $1/N$, and manifestly reproduces local, perturbative supergravity after a straightforward integral transformation. However, this theory is not the full CFT and we expect that the perturbation series is asymptotic at best.

[18] As an example, in AdS_3, N is related to the central charge of the theory. We are then trying to capture the dynamics of a theory with finite c via an infinite-c "free field" description.

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It is worth noting that many features of the collective field description espoused in sec. 5 break a striking resemblance to the matrix models of 2d string theory. This similarity has been an important source of inspiration in the development of the AdS/CFT correspondence [2, 32, 8] and we see it as an interesting guide for future work.

We have argued that it is possible to construct S-matrices and transition amplitudes for spacetime states from the conformal field theory dual. We expect that these amplitudes cannot at the fundamental level be obtained from a local bulk quantum gravity theory, and it would be interesting to learn how the violation of locality is manifested. One approach is to look for breakdown of spacetime locality in deviations of the finite N dynamics from the local-by-construction collective field description proposed in this article. This field theory constructs CFT correlators in a power-series expansion in $1/N$ and g^2_{YM}. Perhaps analyticity of this power series is in itself related to locality. It is worth recalling the usual story where "mean field" assumptions can be destroyed by fluctuations. Perhaps locality and causality are "mean-field" properties. This idea is in line with the suggestions in refs. [33, 6, 7, 34] that local spacetime and causal properties are somehow thermodynamic in character.

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